Topic 7 Gravitational Field

Guiding Questions

- How do two masses interact? Do they need to be in physical contact to do so?
- What do field lines represent for gravitation fields?
- How can we understand the motion of planets and satellites?

E-book

To help you learn this topic better through visualization and self-exploration, an e-book (which can be run on android or IOS smartphones or tablets) is created and can be downloaded through LMS or using:

[Android]
http://iwant2study.org/ospsg/index.php/interactive-resources/physics/02-newtonian-mechanics/08-gravity/153-epub3-gravity

[IOS]
http://iwant2study.org/lookangejss/epub3/20160526gravity.epub

Learning Outcomes (LOs)

Gravitational force between point masses (vector)

a. Recall and use Newton's law of gravitation in the form $F = \dfrac{Gm_1m_2}{r^2}$.

Gravitation field (vector)

b. Show an understanding of the concept of a gravitational field as an example of field of force and define the gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point.

c. Show an understanding that near the surface of the Earth, g is approximately constant and equal to the acceleration of free fall.

d. Derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \dfrac{GM}{r^2}$ for the gravitational field strength of a point mass.

e. Recall and apply the equation $g = \dfrac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems.

f. * Recognise the analogy between certain qualitative and quantitative aspects of gravitational and electric fields *(to be done in the topic of Electric Field)*

Gravitational potential (scalar)

g. Define the gravitational potential at a point as the work done per unit mass in bringing a small test mass from infinity to that point.

h. Solve problems using the equation $\phi = -\dfrac{GM}{r}$ for the gravitational potential in the field of a point mass.

Circular orbits

i. Analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

j. Show an understanding of geostationary orbits and their application.

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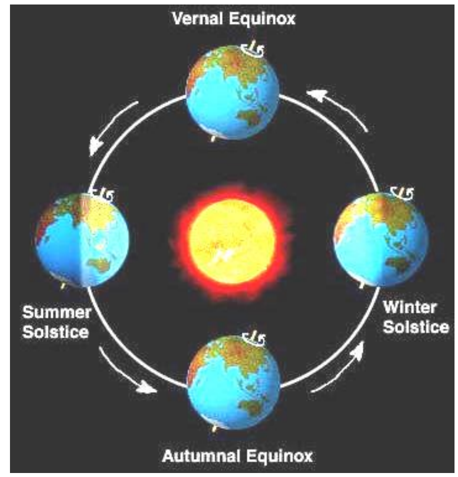
7.1 Introduction

In the topics of Dynamics, Forces, etc, we have been dealing with weights of different objects. Weight is a name given to the force acting on the object due to gravity. Do you know why the weight of an object is different when it is placed at different locations of the Earth surface?

Applications and relevance to daily life

Gravitational force is a force that is evident in our everyday lives and plays a crucial role in many processes on Earth. For instance, the falling of objects when released is caused by the gravitational pull of the Earth. The ocean tides are caused by the gravitational attraction of both the Moon and Sun on the earth's oceans. In terms of planetary motion, gravitational force is responsible for keeping the Earth in its orbit around the Sun, which in turns gives rise to four seasons in some countries, as shown on the right.



7.2 Newton's Law of Gravitation

LO (a)

Gravitation is a natural phenomenon by which physical objects attract each other due to their masses. This force occurs whenever *masses* are present and interestingly, the two objects need not to be in contact with each other for the attraction to take place. It is also worth noting that gravitational force is the weakest of the fundamental forces of nature.

In 1687, Sir Isaac Newton concluded that this non-contact gravitational force must be as responsible for the falling of an apple from a tree, as it is the cause for the rotation of the moon about the earth. Hence he published the Newton's Law of Gravitation which states that:

The mutual force of attraction between any two point masses* is directly proportional to the product of their masses and inversely proportional to the square of the separation between their centres.

** Note that "point mass" refers to a physically small point where the mass seems to concentrate at. For a spherical body, the point is at the centre of the sphere. The size of this small point is negligible as compared to the separation between the bodies.*

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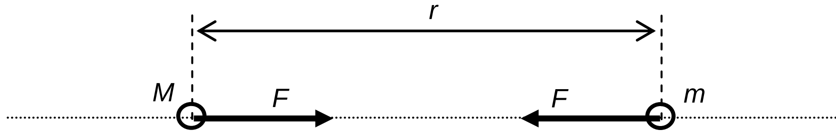


This means that for two point masses M and m, separated by distance r, the magnitude of the gravitational force attracting them towards each other is

$$\boxed{F = \frac{GMm}{r^2}}$$

where

F is the magnitude of the gravitational force [N]

G, the constant of universal gravitation, is 6.67 x 10^{-11} N m^2 kg^{-2}

M is one of the point mass [kg]

m is the other point mass [kg]

r is the ***centre–to–centre distance*** between the two point masses [m]

Important note:

1. r is measured from the centre of object to the centre of the other object. Do not confuse r with the radius of orbit, as they may not be the same!
2. This formula is an example of the ***inverse square law***.

Inquiry:

1) What can you conclude about the two forces in the above diagram?

The two forces in the diagram are ***action-reaction pair*** because each force is acting on the object by the other object and they are of the same type of force.

2) In that case, when Earth pulls you down, why did you not pull Earth up?

You did! But the mass of Earth is relatively much bigger than your mass and hence its acceleration towards you is relatively much smaller.

Example 1

(a) Calculate the gravitational force exerted between the Earth and its Moon, given mass of the Earth, $M_E = 6.0 \times 10^{24}$ kg; mass of the Moon, $M_M = 7.4 \times 10^{22}$ kg; distance between the centres of the Earth and Moon, $D = 3.8 \times 10^8$ m.

Gravitational force: $|F| = \frac{GMm}{r^2} = \frac{(6.67\times10^{-11})(6.0\times10^{24})(7.4\times10^{22})}{(3.8\times10^{8})^2}$ = 2.05 x 10^{20} N

(b) Estimate the gravitational force exerted between you and your nearest neighbour in this lecture theatre.

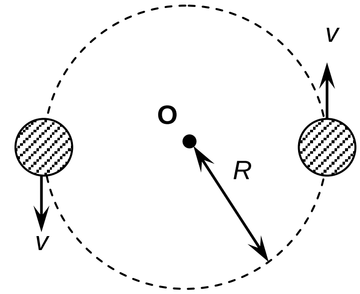
Assume that both masses are 50 kg each and the distance between the centres of gravity of the two persons is 0.50 m.

Magnitude of the force = $|F| = \frac{GMm}{r^2} = \frac{(6.67\times10^{-11})(50)(50)}{(0.50)^2}$ = 6.67×10^{-7} N

(Note: This force is smaller than 0.01% of the weight of an A4-size paper!)

Example 2 (N84/P2/Q7) Binary Stars with same radius of orbit

Two stars of equal mass M move with constant speed v in a circular orbit of radius R about their common centre of mass **O**. What is the net force on each star?



A GM^2/R^2 **B** $GM^2/4R^2$ **C** zero

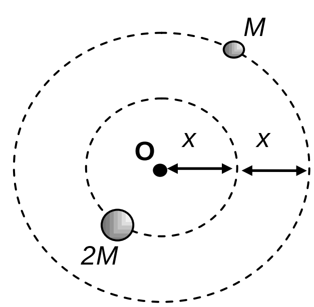
D $2Mv^2/R$ **E** $Mv^2/2R$

The net force acting on each star is the gravitational force.

$F = \frac{GMM}{(2R)^2} = \frac{GM^2}{4R^2}$; Ans: **B**

Example 3 (N09/I/16) Binary Stars with different radii of orbit

Two stars of mass M and $2M$, at a distance $3x$ apart, rotate in circles about their common centre of mass **O**.



The gravitational force acting on the stars can be written as $\frac{kGM^2}{x^2}$.

What is the value of k?

A 0.22 **B** 0.50 **C** 0.67 **D** 2.0

$F = \frac{G(2M)(M)}{(3x)^2} = \frac{2GM^2}{9x^2}$ $\Rightarrow k = 2/9 = 0.22$

Ans: **A**

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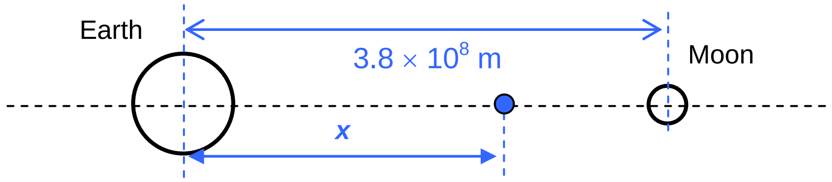
Example 4

A space capsule is travelling between the Earth and its Moon. Considering only the gravitational forces of the Earth and the Moon, determine the position (in terms of the distance from the centre of the Earth) where the capsule experiences zero net gravitational force. (Given: Mass of the Earth, M_E = 6.0×10^{24} kg; mass of the Moon, M_M = 7.4×10^{22} kg; distance between the centres of the Earth and Moon, D = 3.8×10^8 m)



Let the required distance from Earth be x.

gravitational force exerted on capsule by Earth (F_{CE}) = gravitational force exerted on capsule by Moon (F_{CM})

$$\therefore \frac{GM_E m}{x^2} = \frac{GM_M m}{(3.8 \times 10^8 - x)^2}$$

$$\frac{(3.8 \times 10^8 - x)^2}{x^2} = \frac{M_M}{M_E}$$

$$\frac{(3.8 \times 10^8 - x)}{x} = \sqrt{\frac{M_M}{M_E}}$$

$$x = 3.4 \times 10^8 \text{ m}$$

7.3 Gravitational Field

LO (b)

Think about it: How can two objects exert attractive force on each other when they are not in contact with each other?

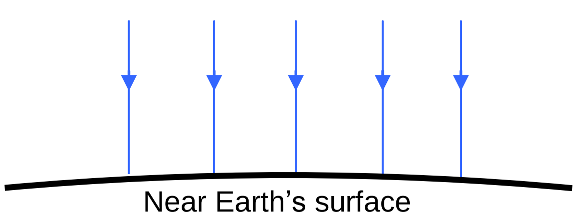
Every object sets up a gravitational field around itself due to its mass. When two objects enter each other's gravitational fields, they will be attracted towards each other. Hence, a gravitational field (which is an example of force fields) is a region of space in which any object lies in it experiences a gravitational force towards the object that creates the field, due to its mass. *(For your information, magnetic fields and electric fields are also examples of force fields.)*

LO (c)

Inquiry:

Gravitational field is invisible and is represented by imaginary field lines. How would the Earth's gravitational field (both near to Earth and over large distances from Earth) looks like?

1) *Draw a few small masses (using pencil) placed **near** the Earth's surface below and draw the direction of gravitational forces acting on them by Earth.*



- The gravitational field near Earth's surface is uniform.
- The closer the field lines, the stronger the field strength.
- The field lines should be drawn parallel to each other and of equal spacing.

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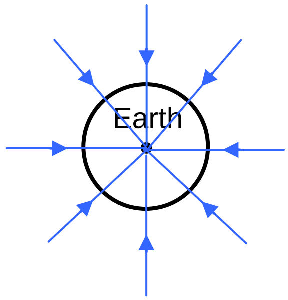
2) *Draw a few small masses (using pencil) placed* ***over large distances*** *from the Earth below and draw the direction of gravitational forces acting on them by Earth.*



- The gravitational field around Earth is non-uniform.
- The field lines should be drawn radially pointing towards the centre of Earth.
- The field lines get further apart (field strength decreases) as it gets further from Earth.

7.4 Gravitational Field Strength (symbol: *g* and units: N kg^{-1} or m s^{-2})

As seen in section 7.3, the gravitational field strength acting on an object decreases (illustrated by an increase in the field line spacing) as the object moves further away from Earth. This means that field strength varies with distance from the source mass (which is the Earth, in this case).

Definition: The gravitational field strength, *g* at a particular point in the gravitational field is **defined** as the

gravitational force per unit mass acting on a small test mass placed at that point.

Inquiry:

Why must the test mass be small?

The test mass must be physically small so that it does not distort or change the gravitational field generated by the source mass.

LO (d)

Based on (i) Newton's law of gravitation, where $|F| = \frac{G\,M\,m}{r^2}$ (gravitational force acting on a point mass, *m* by the source mass, *M*); and (ii) gravitational field strength, *g* is the gravitational force, *F* per unit mass acting on the small test mass, *m*, we may derive that the gravitational field strength,

$$g = \frac{F}{m}$$

$$= \frac{G\,M\,m}{r^2} \Big/ m$$

$$= \frac{GM}{r^2}$$

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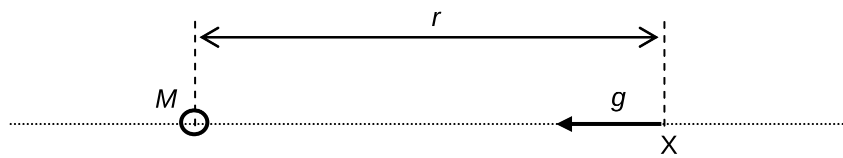


Consider a point X in the field set up by mass *M* located at a distance *r* from the centre of the mass *M*, the magnitude of the gravitational field strength at point X due to mass *M* is

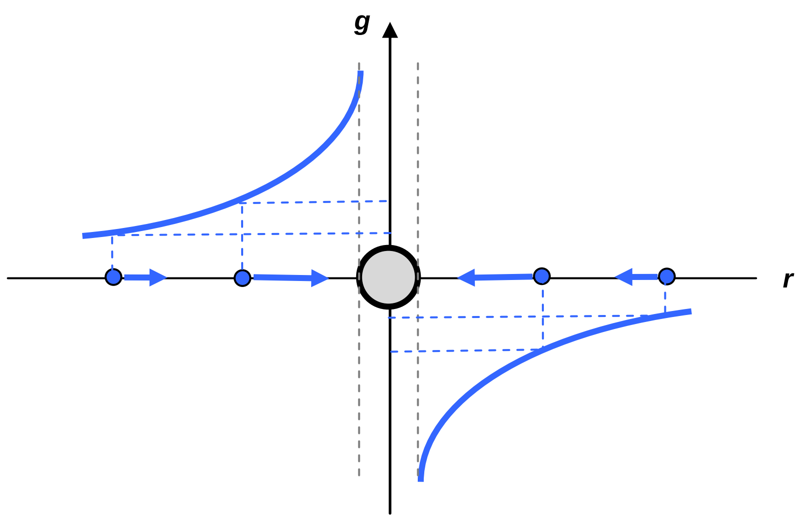
$$\boxed{g = \frac{GM}{r^2}}$$

where

g is the magnitude of the gravitational field strength at point X [N kg^{-1}]
G, the constant of universal gravitation, is 6.67 x 10^{-11} N m^2 kg^{-2}
M is the point mass that generates the gravitational field [kg]
r is the distance from the centre of the point mass M to point X [m]

Note:

1) Gravitational field strength, *g* is a ***vector*** quantity, and it is in **the same direction as the gravitational force**.
2) The gravitational field strength of Earth is approximately constant at 9.81 N kg^{-1}, near its surface. It is also known as the *acceleration of free fall* or the *acceleration due to gravity.*
3) As shown in the derivation above, the gravitational field strength, *g* of the source mass, *M* is independent of the mass of the test mass.
4) As distance *r* of the test mass from source mass increases, *g* decreases in an inverse square law manner. Hence gravitational field is also known as an **inverse square law field**.
5) Graphical representation of gravitational field strength *g* vs distance *r*:



- Take the direction to the right as positive.
- On the left side of the 500-kg mass, the gravitational field strength points to the right (positive *g* values).
- On the right side of the 500-kg mass, the gravitational field strength points to the left (negative *g* values).
- As r increases, magnitude of *g* decreases.

6) Complete section **(A)** & **(B)** in **ICT inquiry worksheet** to strengthen your conceptual understanding on *gravitational field strength* by the next lecture.

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LO (e)

Example 5 (Application of gravitational field strength of one body)

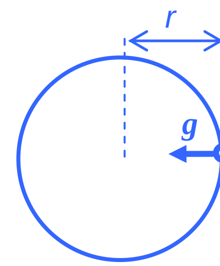
Considering the gravitational field strength on the surface of Earth as 9.81 N kg^{-1}, calculate the average density of the Earth. State any assumptions made in your calculation. (Given: radius of the Earth = 6.37 x 10^{6} m; G = 6.67 x 10^{-11} N m^{2} kg^{-2})

Since $g = \dfrac{GM}{r^2}$, $\therefore 9.81 = \dfrac{(6.67 \times 10^{-11})M}{(6.37 \times 10^{6})^2}$

$\therefore M = 5.968 \times 10^{24}$ kg

Since $V = \dfrac{4}{3}\pi r^3 = 1.083 \times 10^{21}$ m^3, density = $\dfrac{M}{V} = 5.51 \times 10^{3}$ kg m^{-3}

Assumption made: The Earth is spherical.



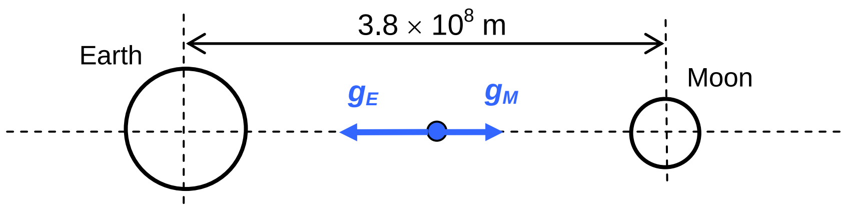
Example 6 (Application of gravitational field strength of multiple bodies)

A space capsule is travelling between the Earth and the moon.

(a) Find the net gravitational field strength at the mid-point between the Earth and the moon.

(b) Hence find the net gravitational force acting on the space capsule at that point.

(Given: Mass of Earth, Moon and space capsule are 6.0 x 10^{24} kg, 7.4 x 10^{22} kg and 100 kg respectively; distance between the centres of the Earth and Moon = 3.8 × 10^{8} m)



(a) Net field strength, g_{net}

$= g_E - g_M$

$= \dfrac{GM_E}{r^2} - \dfrac{GM_M}{r^2}$

$= (6.67 \times 10^{-11})(6.0 \times 10^{24} - 7.4 \times 10^{22})/(1.9 \times 10^{8})^2$

$= 1.09 \times 10^{-2}$ N kg^{-1} (towards the centre of Earth)

(b) Net gravitational force

$= m\, g_{net}$

$= 100\,(1.09 \times 10^{-2})$

$= 1.09$ N (towards the centre of Earth)

7.5 <u>Gravitational Potential Energy</u> (symbol: *U* and units: J)

In the previous topic 5 on *Work, Energy & Power*, we have dealt with the calculation of gravitational potential energy (GPE) using the formula, *mgh*. This formula was applicable in that topic because we were dealing with situations where the height (measured from a certain reference level decided by you) was relatively small (as compared to the radius of Earth) and hence the *g* value of Earth was assumed to be constant (9.81 m s^{-2}) over this height. However if the object is moved through a large height, the assumption that *g* is constant at 9.81 m s^{-2} cannot be valid.

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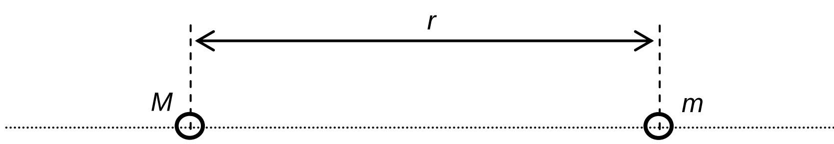
In fact, if you think about it, using *mgh* will determine the 'GPE' of the object from the reference level (decided by you). Hence if the reference level is changed, the 'GPE' value also changes even though the object stays at the same level. The truth is: using *mgh* gives you the *change in GPE* over the height (h), not the actual gravitational potential energy possessed by the object! Hence, to find the gravitational potential energy, U of an object with mass m placed at a distance r away from a source mass M that sets up the gravitational field, physicists arrived at the following equation.



If there are two point masses M and m and they are separated by distance r, the gravitational potential energy U of this system of masses is

$$\boxed{U = -\frac{GMm}{r}}$$

where

U is the gravitational potential energy of the system [J]
G, the constant of universal gravitation, is 6.67×10^{-11} N m^2 kg^{-2}
M is one of the point mass [kg]
m is the other point mass [kg]
r is the ***centre–to–centre distance*** between the two point masses [m]

$$\boxed{U = -\frac{GMm}{r}}$$

<u>To understand the formula for calculating GPE, U:</u>

- Gravitational potential energy is a **scalar** quantity (i.e. it has no direction and a negative U value simply means it is *less than zero*).
- This expression implies that U is **always negative** (less than zero) and the larger the r, the smaller the value of $\frac{GMm}{r}$ and hence the larger the value of $U = -\frac{GMm}{r}$. (For eg. – 2 is larger than – 4)
- When the object is moved to an infinitely far place where $r = \infty$, U becomes zero (which implies maximum gravitational potential energy, since zero is larger than any negative values).
- By standard convention, infinity is taken as the reference level, which has zero gravitational potential energy. But please note that this zero GPE is the maximum U, not the minimum U!

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Definition: The gravitational potential energy, *U* of a point mass placed at a point in the gravitational field is defined as

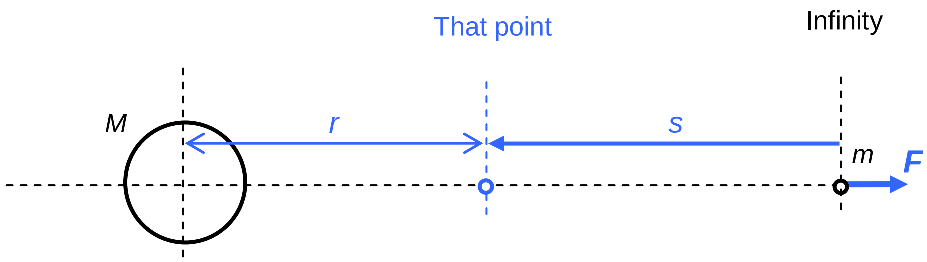
Work done in bringing the <u>point mass</u> from infinity <u>to that point</u>

<u>To appreciate this definition:</u>

- Note that the work done to bring the point mass *m* from infinity (somewhere infinitely far) to a particular point in the field is carried out by an <u>external</u> force (not gravitational force by *M*).

- Since the point mass *m* will be attracted towards the source mass *M* by its gravitational force, the external force acting on mass *m* will be pointing away from *M*, so that mass *m* can be placed/stopped at that particular point.

- Draw the external force *F* and displacement *s* in the diagram below.



- Is the work done positive or negative? Your answer: Negative

- Now link your answer to the sign in the expression $U = -\frac{GMm}{r}$ and its definition above.

- Explain why is there a negative sign in the expression $U = -\frac{GMm}{r}$.

The negative sign of the expression indicates that the work done by the external force is negative as it acts against the attractive nature of the gravitational force.

- Suggest and explain whether the magnitude of the external force *F* is constant as the mass *m* is moved from infinity to that point.

No, because the gravitational force increases as *m* gets closer to *M*.

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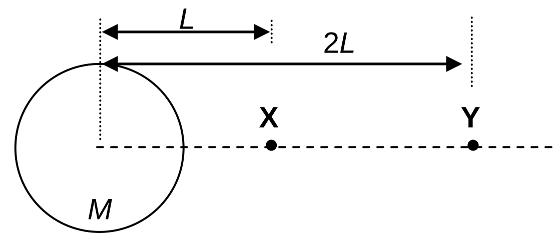
Example 7

The diagram shows two points **X** and **Y** at distances L and $2L$, respectively, from the centre of the Earth (of mass M). Find an expression for the change in the gravitational potential energy of a mass m as it is brought from **X** to **Y**.



Change in GPE, ΔU = Final GPE – Initial GPE

$$= -\frac{GMm}{2L} - \left(-\frac{GMm}{L}\right)$$

$$= \frac{GMm}{2L}$$

Note: The GPE increases as the object is brought away from the source mass (in this case, the Earth).

Escape Velocity - Is it true that "what goes up must come down"?

It is only accurate to say that "what goes up *may* come down". There is a critical velocity at which an object can be launched such that it can escape from Earth permanently. Such a critical velocity is termed as the *escape velocity of Earth*.

Example 8

Determine an expression for the escape velocity, v, of a rocket of mass m launched from the surface of Earth of mass M and radius R.

To escape from Earth, it implies that the rocket must be brought from the *surface of the Earth* to *infinity* where GPE is zero. Thus, the initial KE of rocket must be larger than or at least equal to the change in GPE from Earth surface to infinity.

i.e. $\frac{1}{2}mv^2 \geq [0 - (-\frac{GMm}{R})]$

$\therefore \frac{1}{2}mv^2 \geq \frac{GMm}{R}$

$\therefore v^2 \geq \frac{2GM}{R}$

Hence, $v \geq \sqrt{\frac{2GM}{R}}$

i.e. the escape velocity to reach infinity from Earth is $\sqrt{\frac{2GM}{R}}$.

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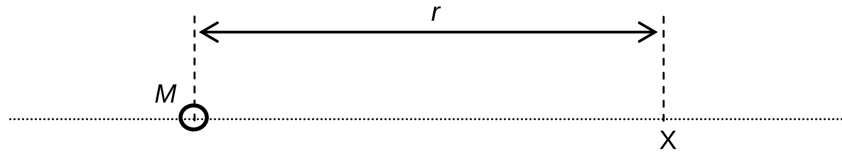
7.6 **<u>Gravitational Potential</u> (symbol: ϕ and units: J kg^{-1})**

LO (g)



Consider a point X in the field set up by mass *M* located at a distance *r* from the centre of the mass *M*, the gravitational potential, ϕ, at point X due to mass *M* is

$$\boxed{\phi = -\frac{GM}{r}}$$

where

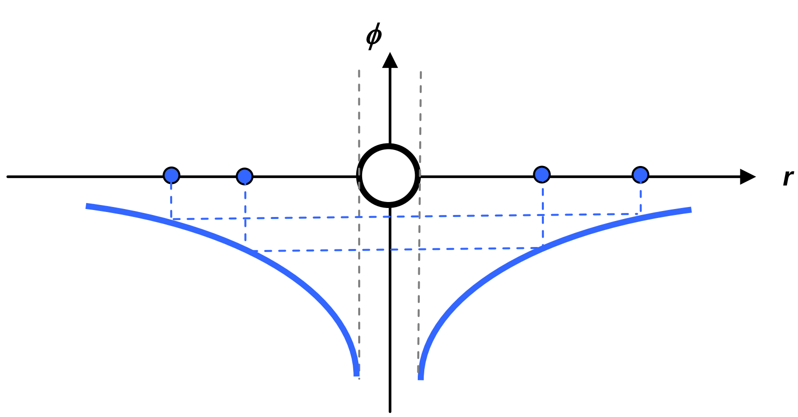
ϕ is the gravitational potential at point X [J kg^{-1}]
G, the constant of universal gravitation, is 6.67×10^{-11} N m^2 kg^{-2}
M is the point mass that generates the gravitational field [kg]
r is the distance from the centre of the point mass M to point X [m]

Definition: The gravitational potential, ϕ, at a point in the gravitational field is defined as

Work done <u>per unit mass</u> in bringing a <u>small test mass</u> from infinity <u>to that point</u>

<u>Note:</u>

1) This expression is similar to the expression for gravitational potential energy, *U*. The only difference from *U* is that ϕ is work done **per unit mass**.
2) Gravitational potential is a **scalar** quantity. (i.e. it has no direction and a negative ϕ value simply means it is less than zero).
3) This expression implies that ϕ is also **always negative** (less than zero) and by convention, the gravitational potential at infinity is also taken to be zero (maximum ϕ value).
4) Similar to gravitational field strength, gravitational potential is also <u>independent of the mass of the test mass.</u>
5) As distance *r* of the point mass from source mass increases, ϕ increases.
6) Graphical representation of gravitational potential ϕ vs distance *r*:



- Gravitational potential value is always negative.
- As r increases, ϕ becomes less negative (increases).

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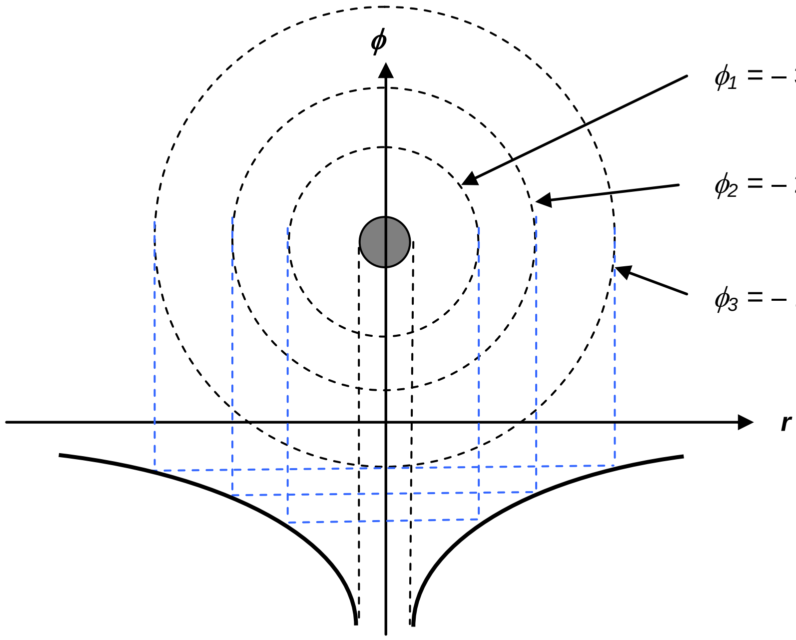
7) Complete section **(C)** & **(D)** in **ICT inquiry worksheet** to strengthen your conceptual understanding on *gravitational potential* by the next lecture.

Equipotential lines are lines that trace positions of equal potential. Examples of 3 equipotential lines around a source mass are shown below in dotted lines.



Note:

- Unlike gravitational field lines that have directions (since g is a vector), equipotential lines have no direction (since ϕ is a scalar).
- Equipotential lines that have equal intervals of potential will not have even spacing, as can be seen from the formula $\phi = -\dfrac{GM}{r}$. The spacing increases with increasing distance r from the mass.

LO (e) & (h)

Example 9 (Application of gravitational field strength and potential)

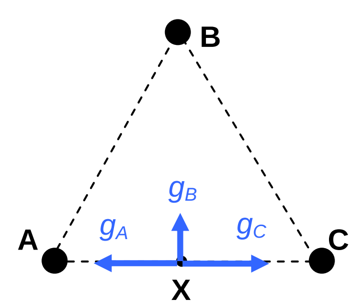
Three identical point masses **A**, **B** and **C**, each of mass 2.0 x 10^5 kg form the vertices of an equilateral triangle with a side length of 1000 m. What is the net gravitational field strength and gravitational potential at point **X** which is located at the midpoint of **AC**?

AX = CX = 1000 / 2 = 500 m BX = $\sqrt{1000^2 - (500)^2} = 866\,\text{m}$

Since g_A will cancel out g_C, net g at X= g_B

$= \dfrac{(6.67 \times 10^{-11})(2.0 \times 10^5)}{(866)^2}$

= 1.78 x 10^{-11} N kg^{-1} (towards B)



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Net gravitational potential at X, ϕ_X

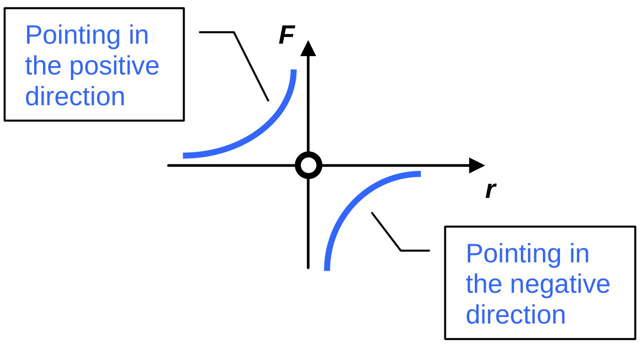
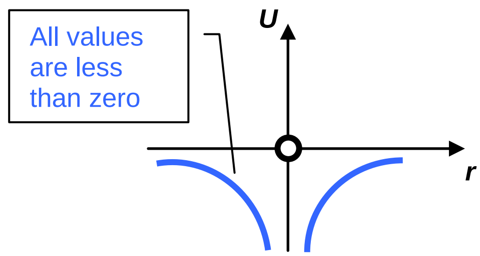
$$= \phi_{XA} + \phi_{XB} + \phi_{XC}$$

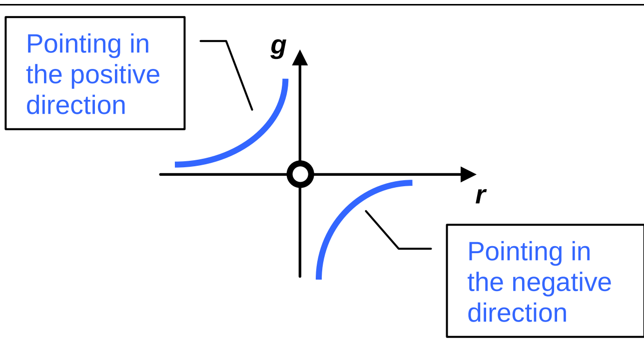
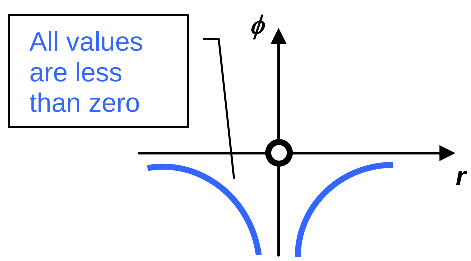
$$= \frac{-GM}{(AX)} + \frac{-GM}{(BX)} + \frac{-GM}{(CX)}$$

$$= -\left(6.67 \times 10^{-11}\right)\left(2.0 \times 10^{5}\right)\left[\frac{1}{(500)} + \frac{1}{(866)} + \frac{1}{(500)}\right]$$

$$= -6.9 \times 10^{-8} \text{ J kg}^{-1}$$

Compare & contrast the four main quantities in Gravitation:

Vector quantities	**Scalar quantities**
Force, $F = \dfrac{GMm}{r^2}$ Unit: N	**Potential energy**, $U = -\dfrac{GMm}{r}$ Unit: J
	

Vector quantities	**Scalar quantities**
Field strength: (Force per unit mass) $g = \dfrac{GM}{r^2}$ Unit: N kg^{-1} (or m s^{-2})	**Potential**: (Potential energy per unit mass) $\phi = -\dfrac{GM}{r}$ Unit: J kg^{-1}
	

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7.7 (a) <u>To understand how *F* is related to *U*:</u>

1) Observe and compare the *F vs r graph* and *U vs r graph* on page 14.
2) If we differentiate $U = -\frac{GMm}{r}$ with respect to *r*, we will get the gradient of the *U* vs *r* graph, $\frac{dU}{dr} = \frac{GMm}{r^2}$. This is the same expression as *F*.
3) However if we observe the two graphs carefully, on the <u>side</u> where the gradient of *U* vs *r* graph is positive, the value of *F* is negative. And on the <u>side</u> where the gradient of *U* vs *r* graph is negative, the value of *F* is positive.
4) Hence it can be concluded mathematically that $\boxed{F = -\frac{dU}{dr}}$

7.7 (b) <u>To understand how *g* is related to ϕ:</u>

5) Similarly, observe and compare the *g vs r graph* and ϕ *vs r graph* on page 14.
6) If we differentiate $\phi = -\frac{GM}{r}$ with respect to *r*, we will get the gradient of the ϕ vs *r* graph, $\frac{d\phi}{dr} = \frac{GM}{r^2}$, which has the same expression as *g.*
7) Again, if we observe the two graphs carefully, on the <u>side</u> where the gradient of ϕ vs *r* graph is positive, the value of *g* is negative. And on the <u>side</u> where the gradient of ϕ vs *r* graph is negative, the value of *g* is positive.
8) Hence, mathematically, $\boxed{g = -\frac{d\phi}{dr}}$
9) Complete section **(E)** in **ICT inquiry worksheet** to strengthen your conceptual understanding on the *relationship between gravitational field strength and potential* by next lecture.

Example 10 (J89/II/2)

Values for the gravitational potential due to the Earth are given in the table below.

Distance from Earth's surface / m	Gravitational potential / MJ kg^{-1}
0	-62.72
390 000	-59.12
400 000	-59.03
410 000	-58.94
Infinity	0

(i) If a satellite of mass 700 kg falls from a height of 400 000 m to the Earth's surface, how much potential energy does it lose?

(ii) Estimate the magnitude of the Earth's gravitational field strength at a height of 400 000 m.

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(i) $\Delta GPE = (m)(\phi_f - \phi_i) = (700)\,[(-62.72) - (-59.03)]\,10^6) = -2.58 \times 10^9$ J
Hence the loss in GPE = 2.58×10^9 J

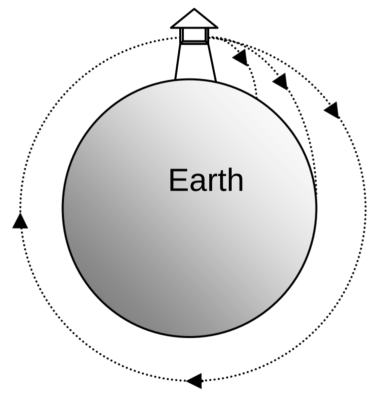
(ii) Since $g = -\dfrac{d\phi}{dr}$,

Consider $|g| = \left|\dfrac{\Delta\phi}{\Delta r}\right|$ for 390 000 m $\le r \le$ 410 000 m

i.e. $|g| = \dfrac{(-58.94\times10^6)-(-59.12\times10^6)}{(410000)-(390000)}$ = 9.0 N kg^{-1}

7.8 Satellite in Circular Orbits

LO (i)



An object projected horizontally near the Earth's surface follows a parabolic trajectory as shown on the right. As the speed of projection increases, the object will reach a speed where the trajectory follows the curvature of the Earth's surface. If air resistance is negligible, the object will orbit round the Earth continuously and will never hit the Earth's surface.

Many man-made satellites move in circular orbits around the Earth. The first man-made satellite, the "Sputnik 1", was launched by Soviet Union in 1957. Since then, hundreds of satellites have been launched into orbit around the Earth. The only force acting on the satellite in a circular orbit is the Earth's gravitational force, which is directed towards the centre of Earth (also the centre of its circular orbit). Since the satellite moves perpendicular to the gravitational force, its magnitude of velocity remains constant while its direction changes. This means that the satellite is travelling in a uniform circular motion (recap Topic 6) with constant distance from the satellite to the centre of Earth.

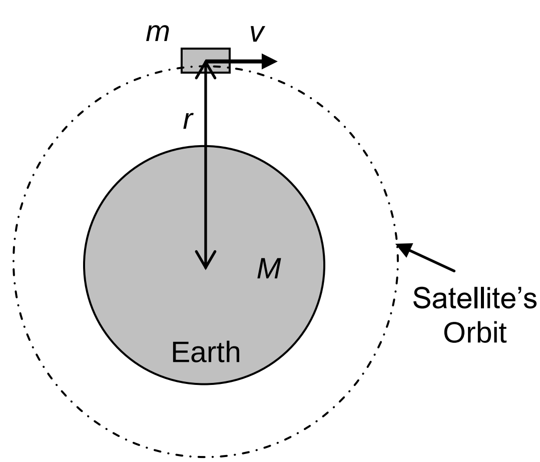
For a satellite (or any object) in circular orbit around a planet (like Earth), the gravitational force acting on the satellite by the planet is the only force that keeps it in circular motion (i.e. the gravitational force is its centripetal force).

Hence, $\Sigma F = \dfrac{mv^2}{r}$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

i.e. $v = \sqrt{\dfrac{GM}{r}}$

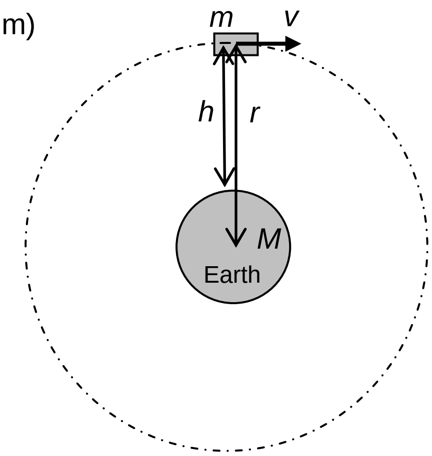
where v denotes the orbiting speed of satellite.



The above formula can be used to calculate the speed required for any object to orbit around a planet of mass M at a constant distance r.

Inquiry:
What will happen to the orbiting satellite if it starts to slow down?
The gravitational force will be higher than the required centripetal force to keep it in the uniform circular motion. Hence the satellite will be pulled closer towards Earth and move in a smaller circular orbit.

Example 11 (Object in orbit)
How fast must the satellite be moving in its circular orbit about the Earth, if it stays at a constant height of three times of Earth's radius, above the Earth's surface?
(Given: mass of Earth = 6.0×10^{24} kg; radius of Earth = 6.4×10^{6} m)



r = 4 x radius of Earth = 2.56×10^{7} m

$$\Sigma F = \frac{mv^2}{r} \rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

thus orbiting speed of satellite,

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})}{(2.56 \times 10^{7})}}$$

i.e. $v = 3954 = 4.0$ km s^{-1}

7.9 Energy of a Satellite in orbit

A satellite in orbit possesses both kinetic energy, E_K, (by virtue of its motion) and gravitational potential energy, E_P, (by virtue of its position within the Earth's gravitational field).

Hence, total energy of an orbiting satellite, $E_T = E_P + E_K$

$$= -\frac{GMm}{r} + \frac{1}{2}mv^2 \qquad \text{--- Equation (1)}$$

Recall that for a satellite in orbit, its gravitational force acts as the centripetal force:

$$\Sigma F = \frac{mv^2}{r} \rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

i.e. $$\boxed{E_K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}} \qquad \text{--- Equation (2)}$$

Substituting equation (2) into (1),

Hence total energy of an orbiting satellite, $E_T = E_P + E_K$

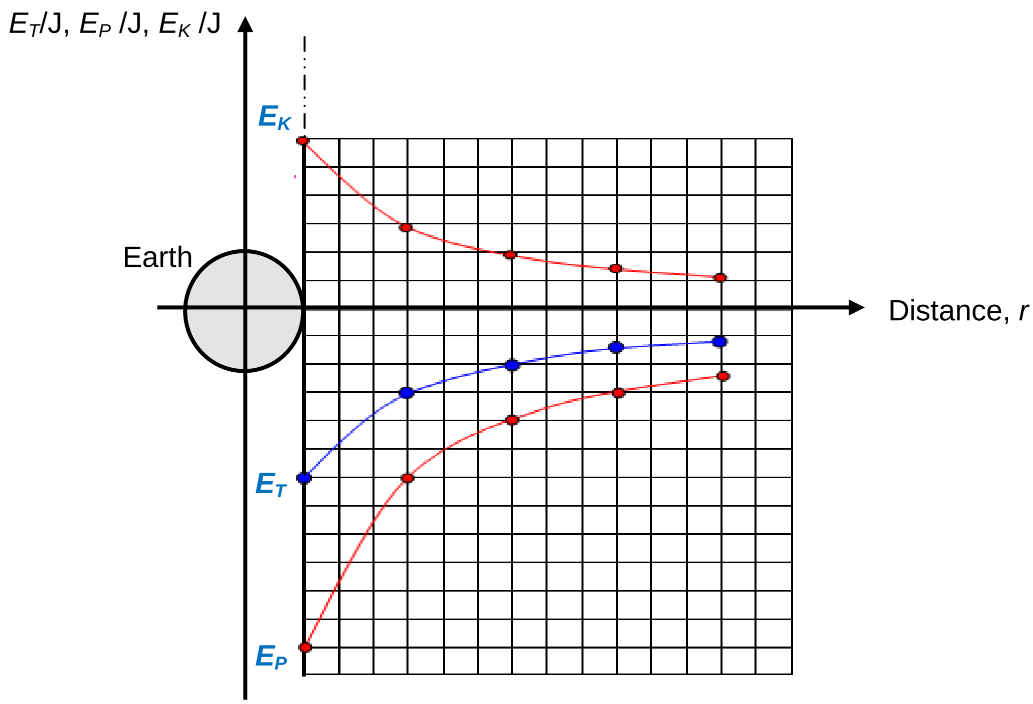
$$E_T = -\frac{GMm}{r} + \frac{1}{2}mv^2$$

$$E_T = -\frac{GMm}{r} + \frac{GMm}{2r}$$

$$\boxed{E_T = -\frac{GMm}{2r}}$$

[Note: $E_T = -E_K$ or ½ E_P]

A typical graph showing the relationship between E_T, E_P and E_K with respect to the distance, r, from centre of Earth, is shown below. Label the graphs accordingly.



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Example 12

An Earth satellite of mass 200 kg lost energy slowly through atmospheric resistance and fell from an orbit of radius 8.0 x 10^6 m to 7.8 x 10^6 m. Calculate the changes in the potential, kinetic and total energies of the satellite as a result of this transition. (Given mass of Earth = 6.0 x 10^{24} kg)

Consider $\Sigma F = \frac{mv^2}{r}$,

$$F_G = \frac{mv^2}{r}$$

i.e. $\frac{GMm}{r^2} = \frac{mv^2}{r}$

$$\frac{1}{2}\frac{GMm}{r} = \frac{1}{2}mv^2$$

Thus, change in kinetic energy, ΔE_K

$$= \frac{GMm}{2r_f} - \frac{GMm}{2r_i}$$

$$= \frac{(6.67\text{x}10^{-11})(6.0\text{x}10^{24})(200)}{2}\left(\frac{1}{7.8\text{x}10^6} - \frac{1}{8.0\text{x}10^6}\right)$$

= 1.28 x 10^8 J (increase)

Since E_P = -2 E_K,

$\therefore$ ΔE_P = -2 ΔE_K

Thus, change in potential energy, ΔE_P = – 2 (1.28 x 10^8)

= – 2.57 x 10^8 J (decrease)

Since $E = \frac{1}{2} E_P$

$\therefore$ $\Delta E = \frac{1}{2} \Delta E_P$

= – 1.28 x 10^8 J (decrease)

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7.10 Kepler's Third Law

Earlier, it was stated that the gravitational force acting on a satellite in orbit is the centripetal force to keep it in circular motion.

i.e. $\Sigma F = mr\omega^2$

$$\Rightarrow \quad \frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

Hence, $T^2 = \frac{4\pi^2}{GM} r^3$

or $\boxed{T^2 \propto r^3}$

This relationship between T and r is known as the **Kepler's Third Law**, which states that the square of the period of an object **in circular orbit** is directly proportional to the cube of the radius of its orbit.

Note:

- The Kepler's Third Law is only applicable to masses in circular orbit, whereby the gravitational force is the only force acting on it to act as its centripetal force.

7.11 Geostationary Satellites

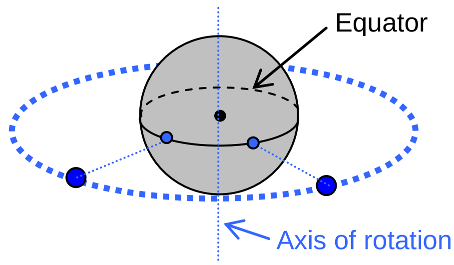
LO (j)

A geostationary (Earth) satellite is a satellite that rotates around the Earth such that it is always positioned above the same point on the Earth's surface. Hence from the point of view of an observer standing at that point on Earth's surface, the geostationary satellite appears to be always 'stationary' above him/her (when actually, both observer and satellite are rotating at the same angular speed). In order for a satellite to be moving in a geostationary orbit, it needs to meet the following conditions:

(1) Geostationary satellites must be **placed vertically above the equator** (so that its axis of rotation is the same as the Earth's);
(2) They must move **from west to east** (so that it moves in the same direction as the rotation of the Earth about its own axis);
(3) The **satellite's orbital period** must be equal to **24 hrs** (so that it is the same as the Earth's rotational period about its own axis).

Inquiry:

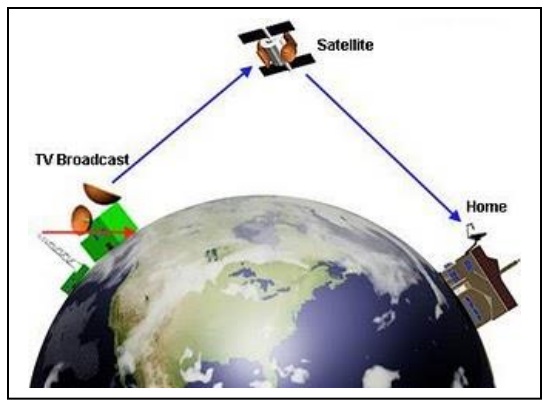
Sketch in the diagram below to illustrate how a geostationary satellite orbits around the Earth.



All geostationary satellites must be placed in an orbit at a fixed distance (around 35 700 km) from the Earth's surface, in order to rotate with the same period as Earth. Do you know why? (Hint: recall Keper's Third Law)

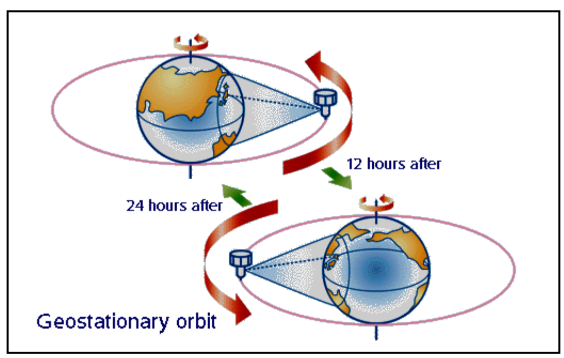
Advantages of geostationary satellites:

1. A geostationary satellite is ideal for telecommunication purposes since it remains 'stationary' above the same spot on the Earth's surface at all times. The distance between the satellite and the transmitting station on Earth is kept relatively constant and a clear line of 'vision' between the transmitter and the receiver allows continuous and uninterrupted signal transmission.



2. Since it is always at the same position relative to the Earth's surface, there is no need to keep adjusting the direction of the satellite dish to transmit or receive signals to or from the geostationary satellite.

3. As geostationary satellites are positioned at a high altitude (a distance of 35 700 km away from the surface of the Earth), it can view and scan a large section of the Earth surface continuously. Hence, they are ideal for meteorological applications and remote imaging.



Disadvantages of geostationary satellites:

1. As geostationary satellites are positioned at such a high altitude, the resolution of the images may not be as good as those captured by the lower orbiting satellites.

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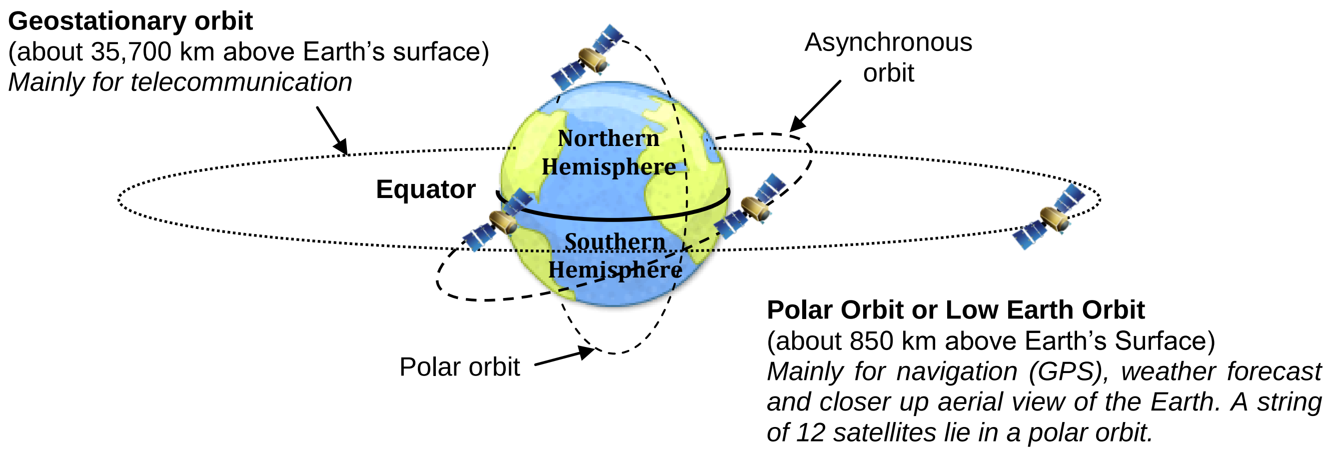
2. Because of its high altitude, there may be a delay in the reception of the signals resulting in a lag time for live international broadcast or video conferencing.

3. The transmitting stations in countries positioned at latitudes higher than 60 degrees may not be able to receive strong signals from geostationary satellites, as the signals would have to pass through a larger amount of atmosphere. This is true for countries beyond the 60 degrees latitude 'belt', both on the northern and southern sides.

Besides geostationary satellites which are placed at a large distance from Earth, there are other types of satellite which orbit at lower altitudes from Earth, like the polar orbit satellites as shown below.



Satellites in polar orbits rotate around the Earth over the poles, in a constant plane perpendicular to the equator. Polar satellites have much lower altitudes (about 850 km) and they serve to provide detailed information about the weather and cloud formation. However satellites in this type of orbit can view only a narrow strip of Earth's surface on each orbit. Strips of images must be "stitched together," to produce a larger view.

Advantages of low altitude orbit satellites (eg. polar orbit):

1. Due to their lower altitudes, these satellites can capture images of the Earth's surface with higher resolution. Polar satellites have the advantage of photographing close-up images of Earth.

2. There is reduced lag time or delay between the transmission and reception of the signal.

Disadvantages of low altitude orbit satellites (eg. polar orbit):

1. It is not possible to view the same spot on the Earth's surface continuously by a single satellite in a polar orbit because of its high speed. A typical low orbit satellite takes only 2 hours to make one revolution round the Earth. In order to have a continuous relay of data, there must be a series or chain of satellites in the same orbit so that one 'takes over' the predecessor's function.

2. Because the satellite changes its location constantly with respect to the Earth's surface, the direction of the satellite dish would need to be adjusted constantly as well.

Example 13

Determine the typical orbital radius of a geostationary satellite around Earth. (Given: mass of Earth = 6.0×10^{24} kg)

The orbital period for geostationary satellite = period of rotation of Earth = 24 hours

Thus, T = 24 hr x 60 min x 60 sec = 8.64×10^4 s

Using $\Sigma F = m r \omega^2$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$\therefore \quad r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\text{x}10^{-11})(6.0\text{x}10^{24})(8.64\text{x}10^4)^2}{4\pi^2}}$$

i.e. $r = 4.2 \times 10^7$ m (about 6.6 times of Earth's radius)

<u>Inquiry:</u>

Would a geostationary satellite that orbit around planet Mars be at the same distance r (= 4.2×10^7 m), as Example 13? Why?

No, because the period of rotation and mass of planet Mars are not the same as Earth's.

Example 14 (J2000/1/8)

Which quantity is not necessarily the same for satellites that are in geostationary orbits around the Earth?

A angular velocity

B centripetal acceleration

C kinetic energy

D orbital period

Kinetic energy is dependent on the satellite's mass and velocity. Hence different satellites of different masses may have different kinetic energies. **Ans: C**

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Example 15

A spacecraft was launched from Earth into a circular orbit around Earth that was maintained at an almost constant height of 189 km from the Earth's surface. Assuming the gravitational field strength in this orbit is 9.4 N kg^{-1}, and the radius of the Earth is 6 370 km,

a) Calculate the speed of the spacecraft in this orbit.

b) Find the time to complete one orbit.

c) Comment whether this spacecraft is in a geostationary orbit.

(a) Since $\Sigma F = \dfrac{mv^2}{r}$, $F_G = \dfrac{mv^2}{r}$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r^2} = \frac{v^2}{r}$$

$$g = \frac{v^2}{r}$$

$$\therefore v = \sqrt{gr}$$

$= [(9.4)(6370 + 189)(10^3)]^{0.5}$

$= 7852$

$= 7.85 \times 10^3$ m s^{-1}

(b) $T = 2\pi r/v = 2\pi(6559 \times 10^3) / 7852$

$= 5249$ s

$= 87.5$ min

(c) Spacecraft is not in geostationary orbit as the period of rotation is less than period of Earth's rotation about its own axis (24 hr).

-- End of note --

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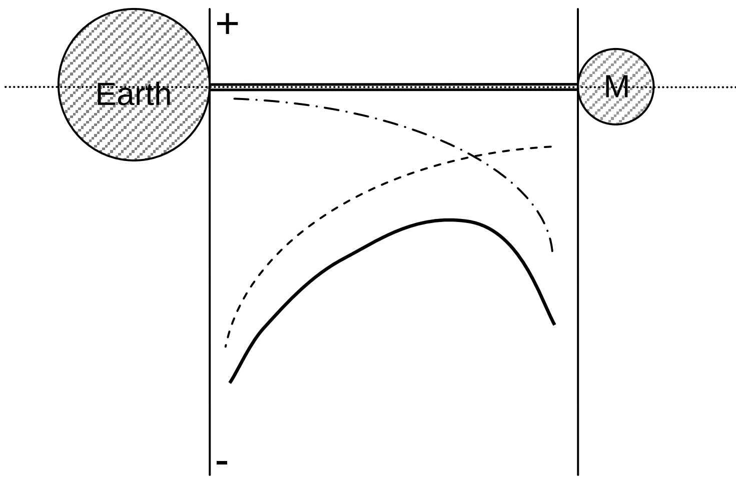
Appendix – Example of Application of $\phi_{total} = \sum \phi_{individual}$

<u>Revision:</u> The total ϕ at a point in a field due to two or more source masses is the scalar addition of the individual ϕ due to each mass at that point, i.e. $\phi_{total} = \sum \phi_{individual}$. The same principle applies when determining U_{total}. The diagram below shows how the gravitational potential varies between the surface of the Moon and the surface of the Earth along the line joining the centres.



Gravitational potential due to **Earth**

Gravitational potential due to **Moon**

Net gravitational potential along the line of centres is equal to the **scalar addition** of the gravitational potentials due to the Earth and Moon.

<u>More food for thought:</u>

Do you know why the moon has no atmosphere?

It is because the speeds of the air particles are higher than their escape speeds and hence they can escape from the moon surface.

If the Sun collapses inwards so that its density increases tremendously, the escape speed will be so large that even light (speed of light = 3.00×10^{8} m s^{-1}) cannot escape from it. The Sun would be invisible. It would become a black hole!

After sitting through the series of lecture on gravitation, can you suggest why most planets are almost spherical? Try looking for the answer in the internet yourself!

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